United Kingdom
Mathematics Trust

Pink Kangaroo

a member of the Association Kangourou sans Frontières

supported by XTX Markets Overleaf

Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

Enquiries about the Pink Kangaroo should be sent to:

UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

☎ 0113 365 1121 enquiry@ukmt.org.uk www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
B	C	E	C	B	E	D	B	A	C	E	B	A	E	D	E	A	B	E	A	B	E	D	D	C

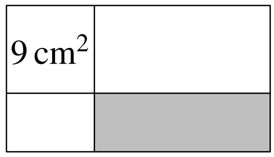
1. The mean age of the members of a jazz band is 21. The saxophonist, singer and trumpeter are 19, 20 and 21 years old respectively. The other three musicians are all the same age. How old are they?

A 21 B 22 C 23 D 24 E 26

Solution **B**

The total of the six ages is $6 \times 21 = 126$. Subtracting 19, 20 and 21 leaves 66 for the total of the three unknown ages. Since these are all equal, each is $66 \div 3 = 22$.

2. A rectangle with perimeter 30 cm is divided by two lines, forming a square of area $9\,\text{cm}^2$, as shown in the figure. What is the perimeter of the shaded rectangle?



A 14 cm B 16 cm C 18 cm D 21 cm E 24 cm

Solution **C**

The square has sides of length $\sqrt{9}\,\text{cm} = 3\,\text{cm}$. Let x and y be the width and length in cm of the shaded rectangle. Then the large rectangle has perimeter $2(3 + x + 3 + y) = 30$. This gives $6 + x + y = 15$ so $x + y = 9$. Hence the perimeter in cm of the shaded rectangle is $2(x + y) = 2 \times 9 = 18$.

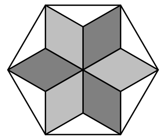
3. The number x has the following property: subtracting $\frac{1}{10}$ from x gives the same result as multiplying x by $\frac{1}{10}$. What is the number x?

A $\frac{1}{100}$ B $\frac{1}{11}$ C $\frac{1}{10}$ D $\frac{11}{100}$ E $\frac{1}{9}$

Solution **E**

From the information given in the question we see that x satisfies the equation $x - \frac{1}{10} = \frac{x}{10}$. Multiplying both sides by 10 gives $10x - 1 = x$, so $9x = 1$. So $x = \frac{1}{9}$.

4. Six congruent rhombuses, each of area $5\,\text{cm}^2$, form a star. The tips of the star are joined to draw a regular hexagon, as shown. What is the area of the hexagon?

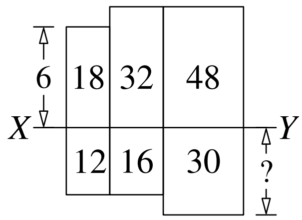


A $36\,\text{cm}^2$ B $40\,\text{cm}^2$ C $45\,\text{cm}^2$ D $48\,\text{cm}^2$ E $60\,\text{cm}^2$

Solution **C**

The six angles in the centre of the hexagon are each $360° \div 6 = 60°$. Therefore the obtuse angles in the rhombuses are each $180° - 60° = 120°$. The obtuse angles in the unshaded triangles are each $360° - 120° - 120° = 120°$. Each unshaded triangle is isosceles with two short sides equal to the side-length of the rhombuses. If we split each rhombus in half along the long diagonal we create isosceles triangles with an angle of $120°$. These are congruent to the unshaded triangles since the two short sides have the same length and the angle between each side is $120°$. Hence the area of each unshaded triangle is half that of a rhombus, namely $2.5\,\text{cm}^2$. The total area is then $(6 \times 5 + 6 \times 2.5)\,\text{cm}^2 = 45\,\text{cm}^2$.

5. Six rectangles are arranged as shown. The number inside each rectangle gives the area, in cm^2, of that rectangle. The rectangle on the top left has height 6 cm.
What is the height of the bottom right rectangle?



A 4 cm B 5 cm C 6 cm D 7.5 cm E 10 cm

Solution **B**

To obtain an area of $18\,\text{cm}^2$, the width of the top left rectangle must be $(18 \div 6)\,\text{cm} = 3\,\text{cm}$. Then the bottom left rectangle must have height $(12 \div 3)\,\text{cm} = 4\,\text{cm}$. Similarly the bottom middle rectangle must have width $(16 \div 4)\,\text{cm} = 4\,\text{cm}$, the top middle rectangle must have height $(32 \div 4)\,\text{cm} = 8\,\text{cm}$, the top right rectangle must have width $(48 \div 8)\,\text{cm} = 6\,\text{cm}$ and the bottom right rectangle must have height $(30 \div 6)\,\text{cm} = 5\,\text{cm}$.

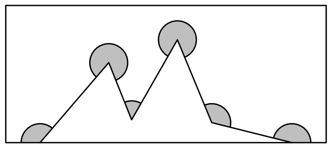
6. How many five-digit positive integers have the product of their digits equal to 1000?

A 10 B 20 C 28 D 32 E 40

Solution **E**

The prime factorisation of $1000 = 2^3 \times 5^3$. To obtain a factor of 5^3 in the product of the digits, three of the digits must be 5 since that is the only one-digit multiple of 5. The other two digits must have product $2^3 = 8$. This can be obtained by using the digits 8 and 1, or the digits 2 and 4. Using 8 and 1, there are five choices for where to place the 8 and then four choices for where to place the 1, hence $5 \times 4 = 20$ possibilities. Similarly there are 20 possibilities using 2 and 4, hence 40 possibilities overall.

7. Five line segments are drawn inside a rectangle as shown. What is the sum of the six marked angles?



A 360° B 720° C 900° D 1080° E 1120°

Solution **D**

The six marked angles, together with the 4 right angles of the rectangle, are the 10 interior angles of a decagon. Since angles in a decagon add up to $(10 - 2) \times 180^\circ = 8 \times 180^\circ$, the six marked angles add up to $(8 \times 180 - 4 \times 90)^\circ = 6 \times 180^\circ = 1080^\circ$.

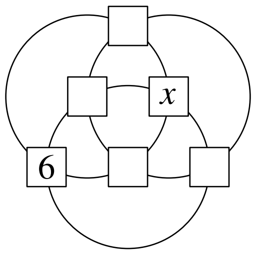
8. At half-time in a handball match, the home team was losing 9–14 to the visiting team. However, in the second half, the home team scored twice as many goals as the visitors and won by one goal.
What was the full-time score?

A 20–19 B 21–20 C 22–21 D 23–22 E 24–23

Solution **B**

Let x be the number of goals scored by the visiting team in the second half, making their final score $14+x$. The home team scored twice as many so their final score was $9+2x$. They won by one goal so $9 + 2x = 14 + x + 1$. Subtracting 9 and x from both sides gives $x = 6$. Substituting this into $9 + 2x$ and $14 + x$ gives the final score as 21-20.

9. The numbers from 1 to 6 are to be placed at the intersections of three circles, one number in each of the six squares. The number 6 is already placed.
Which number must replace x, so that the sum of the four numbers on each circle is the same?



A 1 B 2 C 3 D 4 E 5

Solution **A**

Every number is placed on two circles, so the total of all three circles combined is $2 \times (1 + 2 + 3 + 4 + 5 + 6) = 42$. Thus each circle has total $42 \div 3 = 14$. Hence the three numbers that appear in a circle with the 6 must add to $14 - 6 = 8$. There are only two ways to get a total of 8 from three of the other numbers: $1 + 2 + 5$ or $1 + 3 + 4$. Since x appears on both of the circles with 6, and the only number that appears in both $1 + 2 + 5$ and $1 + 3 + 4$ is 1, we have $x = 1$.

10. Ahmad walks up a flight of eight steps, going up either one or two steps at a time. There is a hole on the sixth step, so he cannot use this step. In how many different ways can Ahmad reach the top step?

A 6 B 7 C 8 D 9 E 10

Solution **C**

The sixth step has a hole, so Ahmad must jump from the fifth to the seventh and then he steps up one step to get to the eighth step. Hence we only need to count the number of ways to get to the fifth step using one or two steps at a time. He could use only "one-steps" (one way), or he could use one "2-step" and three "1-steps" (four ways: 2111, 1211, 1121, 1112), or he could use two "2-steps" and one "1-step" (three ways: 221, 212, 122). Altogether this is $1 + 4 + 3 = 8$ ways.

11. There were five teams entered in a competition. Each team consisted of either only boys or only girls. The number of team members was 9, 15, 17, 19 and 21. After one team of girls had been knocked out of the competition, the number of girls still competing was three times the number of boys. How many girls were in the team that was eliminated?

A 9 B 15 C 17 D 19 E 21

Solution **E**

The total number of team members is $9 + 15 + 17 + 19 + 21 = 81$. Let x be the number of girls in the team eliminated. Then the number of remaining players is $81 - x$ and a quarter of these must be boys (since there remains three times as many girls as boys). The values of $\frac{81-x}{4}$ for $x = 9$, 15, 17, 19 and 21 are 18, 16.5, 16, 15.5 and 15 respectively. The only one of the latter list which equals the number of members of a team is 15 (when $x = 21$) and none of the latter list can be made by a combination of two or more of 9, 15, 17, 19 and 21. Therefore the team eliminated consisted of 21 girls. This left a team of 15 boys and teams of 9, 17 and 19 girls. The total number of girls left was 45 and this was three times the number of boys.

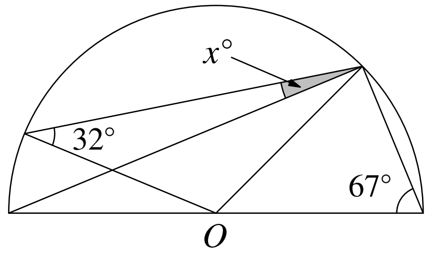
12. Tom had ten sparklers of the same size. Each sparkler took 2 minutes to burn down completely. He lit them one at a time, starting each one when the previous one had one tenth of the time left to burn. How long did it take for all ten sparklers to burn down?

A 18 minutes and 20 seconds
B 18 minutes and 12 seconds
C 18 minutes
D 17 minutes
E 16 minutes and 40 seconds

Solution **B**

Tom let the first nine sparklers burn for nine-tenths of two minutes before lighting the next sparkler. Since $9 \times \frac{9}{10} \times 2 = \frac{162}{10} = 16\frac{2}{10}$, the first nine sparklers burned for 16 minutes and 12 seconds. The last one burned for the full two minutes so the total time was 18 minutes and 12 seconds.

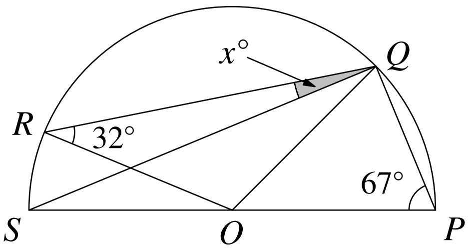
13. The diagram shows a semicircle with centre O. Two of the angles are given. What is the value of x?



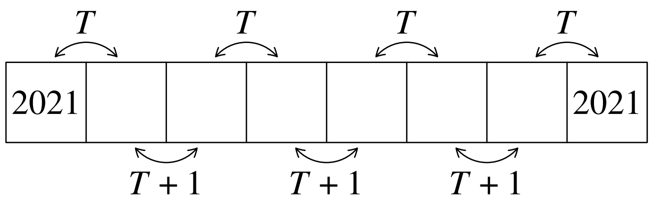
A 9 B 11 C 16 D 17.5 E 18

SOLUTION **A**

Triangle OPQ is isosceles (OP and OQ are both radii), so angle $OQP = 67°$. Angle $PQS = 90°$ (angle in a semicircle). Hence angle $OQS = 90° - 67° = 23°$. Triangle OQR is also isosceles (OQ and OR are both radii) so angle $OQR = 32°$. Hence $x = 32 - 23 = 9$.



14. Each box in the strip shown is to contain one number. The first box and the eighth box each contain 2021. Numbers in adjacent boxes have sum T or $T + 1$ as shown. What is the value of T?

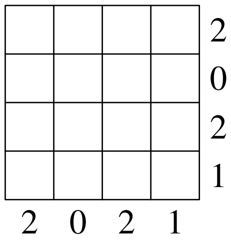


A 4041 B 4042 C 4043 D 4044 E 4045

SOLUTION **E**

Starting on the left, the first two boxes add to T so the second box is $T - 2021$. The second and third boxes add to $T + 1$ so the third box is 2022. Continuing in this way, the numbers obtained are $T - 2022, 2023, T - 2023$, and 2024. The final box is 2021 and the final two boxes have total T. So $T = 2024 + 2021 = 4045$.

15. In the 4×4 grid some cells must be painted black. The numbers to the right of the grid and those below the grid show how many cells in that row or column must be black.
In how many ways can this grid be painted?

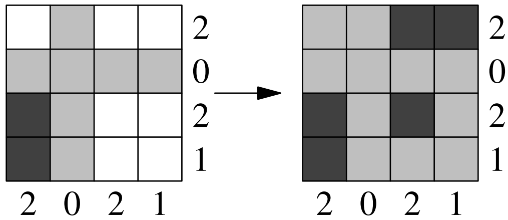


A 1 B 2 C 3
D 5 E more than 5

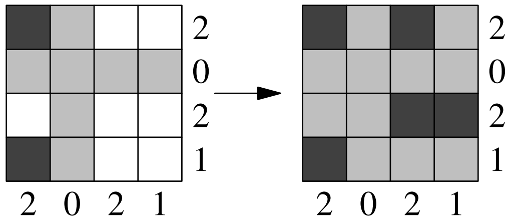
Solution **D**

The diagrams use grey shading for any cells that cannot be painted black. The left-hand column needs two black cells and this can be done in three ways:

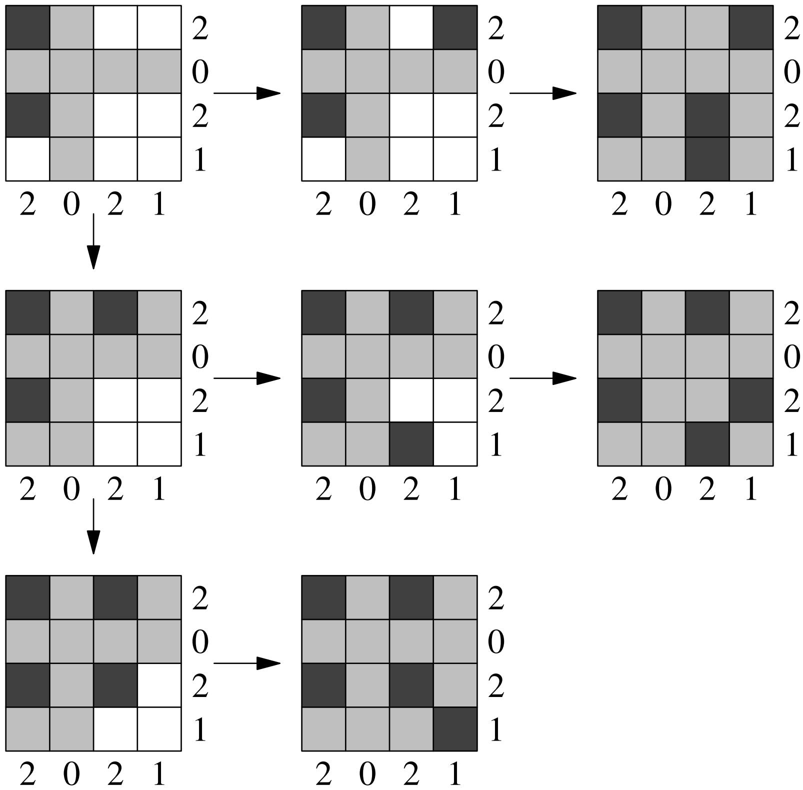
1) Paint the bottom two cells black. Then the top left is grey, forcing the top row to have two black cells on the right. Then we can shade grey the row and column that required one black cell each. This leaves just one possible position for the last black cell.



2) Paint the top and the bottom cells black. The other cell in the column is then grey, leaving the second row up with two black cells on the right. Shade in grey the row and column requiring one black cell. This leaves one position left for a black cell.



3) Paint black the second and fourth cells from the bottom. There are then two options for the second black cell in the top row. Choosing the top right to be black leads to the solution shown on the right. However, colouring the third cell of the top row in black leaves two choices for the second black cell in the third column; each of these leads to the solutions shown.



Thus there are five ways to paint the grid.

16. Five girls ran a race. Fiona started first, followed by Gertrude, then Hannah, then India and lastly Janice. Whenever a girl overtook another girl, she was awarded a point. India was first to finish, then Gertrude, then Fiona, then Janice and lastly Hannah. What is the lowest total number of points that could have been awarded?

A 9 B 8 C 7 D 6 E 5

Solution **E**

Representing the girls by their first initial, and using the left as the front of the race, the initial position is FGHIJ. For I to win, she must at some point overtake F, G and H, earning three points. For G to be second, she must overtake F, earning one point. For J to be fourth, she must overtake H, earning one point. Hence a minimum of five points must be awarded. One possible set of positions during the race could be FGHIJ, FGIHJ, FIGHJ, IFGHJ, IGFHJ, IGFJH, with each change earning one point.

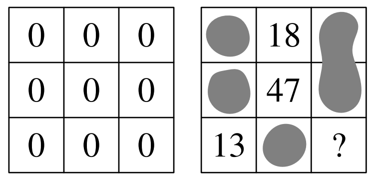
17. The number 2021 has a remainder of 5 when divided by 6, by 7, by 8, or by 9. How many positive integers are there, smaller than 2021, that have this property?

A 4 B 3 C 2 D 1 E none

Solution **A**

For N to have remainder 5 when divided by 6, 7, 8 or 9, we need $N - 5$ to be divisible by 6, 7, 8 and 9. The LCM of 6, 7, 8, 9 is $2^3 \times 3^2 \times 7 = 504$. Hence $N - 5$ is a multiple of 504, so $N - 5 = 0, 504, 1008, 1512, 2016 \ldots$. Then $N = 5, 509, 1013, 1517, 2021 \ldots$. Of these, only the first four are smaller than 2021 and have the stated property.

18. Tatiana's teacher drew a 3×3 grid on the board, with zero in each cell. The students then took turns to pick a 2×2 square of four adjacent cells, and to add 1 to each of the numbers in the four cells. After a while, the grid looked like the diagram on the right (some of the numbers in the cells have been rubbed out.)
What number should be in the cell with the question mark?



A 9   B 16   C 21   D 29   E 34

Solution **B**

Let P be the number of times the top left 2×2 square was picked; and let Q, R, S be the corresponding numbers for the top right, bottom right and bottom left 2×2 squares. The cells in the four corners of the 3×3 grid each appear in exactly one of these 2×2 squares, so $S = 13$ and the number in the cell marked with the question mark is R. The middle top cell is increased $P + Q$ times since it belongs to the top left 2×2 square and the top right 2×2 square. So $P + Q = 18$. The central square belongs to all four 2×2 squares, so $47 = P + Q + R + S = 18 + R + 13 = R + 31$. Hence $R = 16$.

19. Three boys played a "Word" game in which they each wrote down ten words. For each word a boy wrote, he scored three points if neither of the other boys had the same word; he scored one point if only one of the other boys had the same word. No points were awarded for words which all three boys had. When they added up their scores, they found that they each had different scores. Sam had the smallest score (19 points), and James scored the most. How many points did James score?

A 20 B 21 C 23 D 24 E 25

Solution **E**

Let s be the number of times that Sam scored 3 points. If $s \leq 4$ then his maximum score would be $3 \times 4 + 6 = 18$ which is too small. If $s \geq 7$ then his minimum score would be $3 \times 7 = 21$ which is too big. Hence s is either 5 or 6.

Suppose $s = 6$. So Sam had 6 words not shared with the other boys, giving him 18 points. To get a score of 19, he must have shared one word with one other boy and picked three words shared with them both. That means that the other two boys also scored 0 points for those three words. To score more than 19 they must have scored 3 points for each of their 7 remaining words. However one of them has shared a word with Sam so scored a maximum of 19 points. Hence $s \neq 6$.

Hence $s = 5$, giving Sam 15 points for these 5 words. To score 19, Sam must have scored 1 point for another four words and there must have been one word shared by all three boys. Each of the other two must have scored 3 points at least 6 times in order to score more than Sam. Also each scored 0 points for one word. That determines the scores for seven of each boy's words. Between the two, they shared a word with Sam 4 times; and these could be divided between them either 2 each or 1 and 3. The former case leaves only one word each to fix and this could be either 1 point each or 3 points each. But that would give the two boys the same score. Therefore one boy had 3 words shared with Sam and so a score of 21. The other had only one word shared with Sam and the remaining two words must have scored 3 points, giving a total of 25. So James scored 25 points.

20. Let N be the smallest positive integer such that the sum of its digits is 2021. What is the sum of the digits of $N + 2021$?

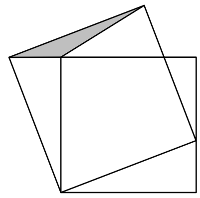
A 10 B 12 C 19 D 28 E 2021

Solution **A**

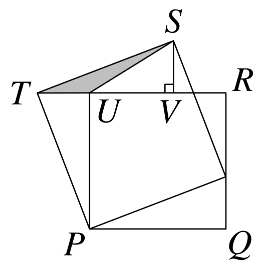
For N to be the smallest integer with digit sum 2021, it must have the least number of digits possible, hence it has as many digits 9 as possible. $2021 = 9 \times 224 + 5$ so N is the 225-digit number $599\ldots999$. Thus $N + 2021$ is the 225-digit number $600\ldots002020$ which has digit sum $6 + 2 + 2 = 10$.

21. The smaller square in the picture has area 16 and the grey triangle has area 1.
What is the area of the larger square?

A 17 B 18 C 19 D 20 E 21



Solution **B**



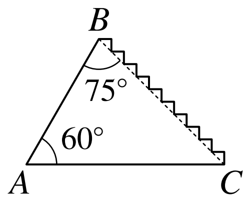
Let V be the foot of the perpendicular dropped from S to TR. Angle STV = angle TPU since both are equal to $90°-$ angle PTU. So triangles STV and TPU are congruent as both contain a right angle and $PT = ST$. Hence $TU = SV$.

The area of the shaded triangle is $\frac{1}{2} \times TU \times SV = \frac{1}{2} \times TU \times TU = 1$, so $TU = \sqrt{2}$.

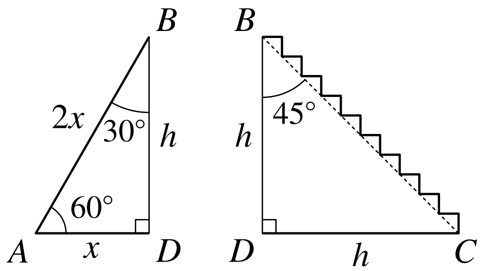
The area of the smaller square is 16 so $PU = 4$. Applying Pythagoras' Theorem to triangle PTU gives $PT^2 = PU^2 + TU^2 = 16 + 2 = 18$. Hence the area of the larger square is 18.

22. A caterpillar crawled up a smooth slope from A to B, and crept down the stairs from B to C. What is the ratio of the distance the caterpillar travelled from B to C to the distance it travelled from A to B?

A 1:1 B 2:1 C 3:1 D $\sqrt{2} : 1$
E $\sqrt{3} : 1$



Solution **E**



Let h be the height of the slope. By dropping the perpendicular from B to the base AC, we create two right-angled triangles ABD and BCD. Angle ABD = $(180 - 60 - 90)° = 30°$ so triangle ABD is half of an equilateral triangle and length AB is twice AD. Let length $AD = x$ so that $AB = 2x$. Then by Pythagoras, $h^2 = (2x)^2 - x^2 = 3x^2$ so $h = \sqrt{3}x$. In triangle BCD the angles are 90°, 45°, 45° so BCD is isosceles and the base $CD = BD = h$.

The total vertical height of the steps is equal to $BD = h$; the horizontal parts of the steps have total length equal to $CD = h$. The ratio of the distance travelled from B to C to the distance travelled from A to B is $2h : 2x = 2\sqrt{3}x : 2x = \sqrt{3} : 1$.

23. A total of 2021 balls are arranged in a row and are numbered from 1 to 2021. Each ball is coloured in one of four colours: green, red, yellow or blue. Among any five consecutive balls there is exactly one red, one yellow and one blue ball. After any red ball the next ball is yellow. The balls numbered 2 and 20 are both green. What colour is the ball numbered 2021?

A Green B Red C Yellow D Blue
E It is impossible to determine

Solution **D**

Each set of 5 consecutive balls must contain exactly one red, one yellow and one blue, and hence must also contain exactly two green balls. The set of 5 consecutive balls that starts with the ball numbered N has the same colours as the set starting with $N + 1$ (and both sets contain the four balls $N + 1, N + 2, N + 3, N + 4$), so the colour of ball $N + 5$ must be the same as ball N. The 20th ball is green, so the 15th, 10th and 5th balls are also green. Hence the first five balls are coloured ?G??G. But each red is immediately followed by a yellow, so the pair in between the two greens are red, yellow. Therefore the first 5 balls are BGRYG. The 2021st ball is the same colour as the 1st ball since $2021 - 1$ is a multiple of 5 and so it is blue.

24. Each of the numbers m and n is the square of an integer. The difference $m - n$ is a prime number.
Which of the following could be n?

A 100 B 144 C 256 D 900 E 10000

Solution **D**

Since m and n are squares, we can write $m = x^2$ and $n = y^2$ for some positive integers x and y. Then the difference is $m - n = x^2 - y^2 = (x + y)(x - y)$. But $m - n$ is prime and has only two factors (1 and itself). Hence $x - y = 1$ and $x + y$ is prime. Rearranging $x - y = 1$ gives $x = y + 1$, so $x + y = 2y + 1$ is prime. For the options available for $n = y^2$, the values of y are $10, 12, 16, 30, 100$ and $2y + 1$ is $21, 25, 33, 61, 201$ respectively. The only prime in this list is 61, so $y = 30$ and $n = 30^2 = 900$.

25. Christina has eight coins whose weights in grams are different positive integers. When Christina puts any two coins in one pan of her balance scales and any two in the other pan of the balance scales, the side containing the heaviest of those four coins is always the heavier side.

What is the smallest possible weight of the heaviest of the eight coins?

A 8 B 12 C 34 D 55 E 256

Solution **C**

Let the coins in such a set have weights (in grams) a, b, c, d, e, f, g, h in ascending order of weight. Note that, to check a set has the stated property, it is enough to check, for each coin, that it together with the lightest coin is heavier than the pair of coins immediately preceding it. Suppose such a set has the minimal value for h. Then $a = 1$ because, if $a \geq 2$ then a matching set in which each coin's weight has been reduced by 1 would still have the property and have a smaller value for h.

Also $b \geq 2$ and $c \geq 3$ because the weights are distinct integers. Applying the check to d, we get $d + 1 > b + c \geq 2 + 3 = 5$. Hence $d \geq 5$. Similarly, $e + 1 > c + d \geq 3 + 5 = 8$, hence $e \geq 8$. Continuing in this way we end up with $h \geq 34$. Moreover, the argument shows that the set of weights 1, 2, 3, 5, 8, 13, 21, 34 does have the required property. Hence the minimum value for the heaviest coin is 34 grams.